Hyperfinite product factors, II.

by

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1. **Introduction.** One of the deep open problems concerning factors of type II_∞ is whether the hyperfinite ones are all isomorphic to $\mathcal{F} \otimes \mathcal{B}(\kappa)$, where $\mathcal{F}$ is the hyperfinite II_1-factor and $\mathcal{B}(\kappa)$ all bounded operators on a separable Hilbert space κ. In [4] we introduced the concept of product factors, which in the hyperfinite case was equivalent to that of ITPFI-factors (i.e. infinite tensor products of finite type I-factors). A factor is said to be a product factor if every normal state ω of $\mathcal{R}$ is asymptotically a product state, i.e. given a finite type I factor M in $\mathcal{R}$ and $\varepsilon > 0$ there is a finite type I factor N such that $M \subset N \subset \mathcal{R}$ and such that $\|\omega - \omega|N \otimes \omega|N^c\| < \varepsilon$. In the present paper we shall show that a factor $\mathcal{R}$ of type II_1 (resp. II_∞) is *-isomorphic to $\mathcal{F}$ (resp. $\mathcal{F} \otimes \mathcal{B}(\kappa)$) if and only if $\mathcal{R}$ is a countably generated product factor (Theorem 4.3). This result is then a characterization of $\mathcal{F}$ and $\mathcal{F} \otimes \mathcal{B}(\kappa)$ in terms of their pre-duals.

2. <u>An inequality.</u> If $\mathcal{R}$ is a von Neumann algebra with a normal semi-finite trace τ we let for an operator A in $\mathcal{R}$. $\|A\|_1 = \tau(|A|)$, where $|A| = (A^*A)^{\frac{1}{2}}$, and $\|A\|_2 = \tau(A^*A)^{\frac{1}{2}}$. Then $\| \ \|_1$ and $\| \ \|_2$ are unbounded semi-norms on $\mathcal{R}$ such if τ is faithful they define norms on $\mathfrak{m}$ and $\mathfrak{m}^{\frac{1}{2}}$ respectively, where $\mathfrak{m}$ is the ideal of definition of τ. We shall need an inequality which relates the two semi-norms. It is together with its proof, an extension of the same inequality proved for $\mathcal{B}(\kappa)$ in [2, Lemma 4.1]:

<u>Lemma 2.1.</u> Let $\mathcal{R}$ be a von Neumann algebra with a normal semi-finite trace τ. Let A and B be positive operators in $\mathcal{R}$. Then

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2 \leq \|A - B\|_1 .$$

<u>Proof.</u> If $C,D \in \mathcal{R}$ with C positive in $\mathfrak{m}$ and D self-adjoint we have

$$\tau(C|D|) \geq |\tau(CD)|. \tag{1}$$

Indeed, $D = D_1 - D_2$ with $D_i \geq 0$ in $\mathcal{R}$ and $D_1D_2 = 0$. Thus we have

$$\tau(C|D|) = \tau(C(D_1+D_2)) = \tau(CD_1) + \tau(CD_2)$$
$$\geq |\tau(CD_1) - \tau(CD_2)| = |\tau(CD)|.$$

Now let $A,B \in \mathcal{R}^+$, the positive part of $\mathcal{R}$. Let $S = A^{\frac{1}{2}} - B^{\frac{1}{2}}$, $T = A^{\frac{1}{2}} + B^{\frac{1}{2}}$. Then $T \geq \pm S$, and $\frac{1}{2}(ST + TS) = A - B$. If $\|A - B\|_1 = \infty$ the lemma is trivial. Assume $|A - B| \in \mathfrak{m}$. If $\mathcal{R}$ is finite then $\mathfrak{m} = \mathcal{R}$ so $S \in \mathfrak{m}$. Otherwise let π be a $*$-representation of $\mathcal{R}$ annihilating $\mathfrak{m}$. Then $\pi(A) = \pi(B)$, hence $\pi(A^{\frac{1}{2}}) = \pi(A)^{\frac{1}{2}} = \pi(B)^{\frac{1}{2}} = \pi(B^{\frac{1}{2}})$, so that $\pi(S) = 0$. Since this holds for all

*-representations π annihilating $\mathcal{M}$, $S \in \overline{\mathcal{M}}$ - the uniform closure of $\mathcal{M}$. Let E be a spectral projection of S such that $E \leq kS$ for some $k > 0$. Then E is finite. Indeed, since $S \in \overline{\mathcal{M}}$, $E \in \overline{\mathcal{M}}$. Let $S_n \in \mathcal{M}$ be a sequence of self-adjoint operators such that $\lim_n \|S_n - E\| = 0$. Then $\lim_n \|S_n^2 - S_n\| \leq \lim_n(\|S_n - E^2\| + \|E - S_n\|) = 0$, so by spectral theory we may assume S_n is a projection E_n in $\mathcal{M}$. But if $\|E - E_n\| < 1$ then $E \sim E_n$ [3, § 105, Théorème]. Since E_n is finite so is E.

Now choose an orthogonal sequence $\{E_n\}_{n\geq 1}$ of finite spectral projections of S with sum I. Say $E_nS \geq 0$ for $n \in J$ and $E_nS \leq 0$ for $n \in J^c$. Let $\tau_n(C) = \tau(E_nC) = \tau(E_nCE_n)$ for $C \in \mathcal{R}$. Then for $C \in \mathcal{M}$ we have $\tau(C) = \Sigma\tau(E_nC) = \Sigma\tau_n(C)$. Also, since τ is normal and $E_nS^2 \geq 0$ for all n, we have $\tau(S^2) = \Sigma\tau_n(S^2)$. Since $|A - B| \in \mathcal{M}$ so is $|ST + TS|$. Furthermore $E_nTE_n \geq \pm E_nS$ for all n. Thus an application of 1) gives

$$\begin{aligned}
\|A - B\|_1 &= \tau(|A - B|) \\
&= \tfrac{1}{2}\tau(|ST + TS|) \\
&= \tfrac{1}{2}\Sigma\tau_n(|ST + TS|) \\
&\geq \tfrac{1}{2}\Sigma|\tau_n(ST + TS)| \\
&= \Sigma|\tau((E_nS)E_nTE_n)| \\
&= \sum_{n\in J}\tau((E_nS)E_nTE_n) + \sum_{n\in J^c}\tau((-E_nS)E_nTE_n) \\
&\geq \sum_{n\in J}\tau(E_nS^2) + \sum_{n\in J^c}\tau(E_nS^2) \\
&= \tau(S^2) \\
&= \|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2.
\end{aligned}$$

The proof is complete.

Remark 2.2. The inequality above can be used to sharpen and give a different proof of an inequality of Murray and von Neumann [1, Ch. III, § 7, Lemme 4]. They showed that if $\mathcal{R}$ is a finite factor, E a projection in $\mathcal{R}$, $T \in \mathcal{R}$ with $0 \leq T \leq I$, then $\|T^{\frac{1}{2}} - E\|_2 \leq$ $\leq 13 \|T - E\|_2^{\frac{1}{4}}$, where the $\| \ \|_2$-norm is with respect to the normalized trace τ. Since $\tau(A^2) \geq \tau(A)^2$ for all self-adjoint A in $\mathcal{R}$ we have from Lemma 2.1 that

$$\|T^{\frac{1}{2}} - E\|_2^2 \leq \|T - E\|_1 = \tau(|T - E|)$$

$$\leq \tau(|T - E|^2)^{\frac{1}{2}} = \|T - E\|_2 .$$

Thus we have
$$\|T^{\frac{1}{2}} - E\|_2 \leq \|T - E\|_2^{\frac{1}{2}} .$$

3. Hyperfinite factors. If $\mathcal{R}$ is a factor we say $\mathcal{R}$ is hyperfinite if there is a sequence $\{M_n\}_{n \geq 1}$ of finite type I subfactors of $\mathcal{R}$ such that $M_n \subset M_{n+1}$ and such that $\bigcup_{n \geq 1} M_n$ is strongly dense in $\mathcal{R}$ (here and everywhere else we assume $I \in M$ whenever we say M is a subfactor of $\mathcal{R}$). In this section we shall first give an equivalent definition of hyperfinite factors, and then prove some related results. We say a factor $\mathcal{R}$ is countably generated if there is a strongly dense sequence $\{T_i\}$ of operators in $\mathcal{R}$. Then the $*$-algebra generated by the T_i is strongly dense in $\mathcal{R}$, hence by the Kaplansky density theorem there is a sequence which is strongly dense in the positive part of the unit ball in $\mathcal{R}$.

Lemma 3.1. Let $\mathcal{R}$ be a factor acting on a Hilbert space κ. Then $\mathcal{R}$ is hyperfinite if and only if $\mathcal{R}$ is countably generated and has the property that if M is a finite type I subfactor of $\mathcal{R}$, $T \in \mathcal{R}^+$, $x_1, \cdots, x_r \in \kappa$, and $\varepsilon > 0$, we can find a finite type I factor N with $M \subset N \subset \mathcal{R}$ such that there is $S \in N^+$ with $\|S\| \leq \|T\|$ and $\|(S - T)x_k\| < \varepsilon$, $k = 1, \cdots, r$.

Proof. If $\mathcal{R}$ is finite and of type I the lemma is trivial, so we exclude this case. Suppose first $\mathcal{R}$ is hyperfinite. Let $\{M_n\}$ be an increasing sequence of finite type I subfactors of $\mathcal{R}$ whose union is strongly dense in $\mathcal{R}$. Since each M_n is countably generated so is clearly $\mathcal{R}$. Let M be a finite type I subfactor of $\mathcal{R}$. Let $T \in \mathcal{R}^+$, $\varepsilon > 0$, and $x_1, \cdots, x_r \in \kappa$. By [4, Lemma 2] $M^c = M' \cap \mathcal{R}$ is a hyperfinite factor $*$-isomorphic to $\mathcal{R}$. Say $\{N_n\}_{n \geq 1}$ is an increasing sequence of finite type I factors generating M^c. Then $R_n = M \cup N_n$ is an increasing sequence of finite type I factors generating $\mathcal{R}$, and $R_n \supset M$ for all n. By the Kaplansky density theorem [1, Ch.I, § 3, Théorème 3] there exists R_n and $S \in R_n^+$ with $\|S\| \leq \|T\|$ such that $\|(S - T)x_k\| < \varepsilon$, $k = 1, \cdots, r$.

Conversely assume $\mathcal{R}$ is countably generated and has the property in the lemma. If x is unit vector in κ then $[\mathcal{R} x]$ is a nonzero projection in $\mathcal{R}'$, and the map $\mathcal{R} \to \mathcal{R}\,[\mathcal{R} x]$ is an isomorphism. Since $\mathcal{R}$ is countably generated the Hilbert space $[\mathcal{R} x]\kappa$ is separable. Thus we may assume κ is separable. Let $\{x_i\}_{i \geq 1}$ be a dense sequence of vectors in the unit ball of κ. Let $\{T_j\}_{j \geq 1}$ be a dense sequence in the unit ball of $\mathcal{R}^+$ with $T_1 = I$. We shall by induction construct a sequence of finite type I subfactors M_n of $\mathcal{R}$ and operators $S_j^n \in M_n$ such that

1) $M_1 \subset M_2 \subset \cdots$.

2) $\|(S_j^n - T_j)x_k\| < 2^{-n}$, $\quad j, k = 1, \cdots, n$.

3) $0 \leq S_j^n \leq I$, $j = 1, \cdots, n$.

Let $M_1 = \mathbb{C}I$ be the type I_1-subfactor of $\mathcal{R}$ and let $S_1^1 = I$. Assume $M_1, \cdots, M_{n-1}$ together with the operators S_j^k are constructed such that 1), 2), and 3) hold. By hypothesis we can find a finite type I subfactor N_1 of $\mathcal{R}$ such that $M_{n-1} \subset N_1$ and $S_1^n \in N_1$ with $0 \leq S_1^n \leq I$ such that $\|(S_1^n - T_1)x_k\| < 2^{-n}$, $k = 1, \cdots, n$. Now choose a finite type I subfactor N_2 of $\mathcal{R}$ such that $N_1 \subset N_2$ and $S_2^n \in N_2$ such that $0 \leq S_2^n \leq I$ and $\|(S_2^n - T_2)x_k\| < 2^{-n}$, $k = 1, \cdots, n$. Continue this procedure until we have found finite type I subfactors N_j of $\mathcal{R}$ such that $N_n \supset N_{n-1} \supset \cdots \supset N_1 \supset M_{n-1}$ and $S_j^n \in N_j$ such that $0 \leq S_j^n \leq I$ and $\|(S_j^n - T_j)x_k\| < 2^{-n}$, $k = 1, \ldots, n$. Letting $M_n = N_n$ we have completed the induction argument and thus constructed the sequence $\{M_n\}$ such that 1), 2), and 3) hold.

We next show that $\bigcup_{n \geq 1} M_n$ is strongly dense in $\mathcal{R}$. Let $\varepsilon > 0$, $y_1, \cdots, y_r \in \kappa$ and $T \in \mathcal{R}$. There is no restriction to assume that $0 \leq T \leq I$. Choose T_j such that $\|(T_j - T)y_k\| < \frac{\varepsilon}{3}$ for $k = 1, \cdots, r$. Choose x_{i_k}, $k = 1, \cdots, r$, in the sequence $\{x_i\}$ such that $\|x_{i_k} - y_k\| < \frac{\varepsilon}{6}$. Let $n \geq j$ be a positive integer such that $2^{-n} < \frac{\varepsilon}{3}$ and such that $n \geq \max\{i_k : k = 1, \cdots, r\}$. Choose $S_j^n \in M_n$ with $0 \leq S_j^n \leq I$ such that $\|(S_j^n - T_j)x_i\| < 2^{-n}$, $i = 1, \cdots, n$. Then in particular $\|(S_j^n - T_j)x_{i_k}\| < \frac{\varepsilon}{3}$ for $k = 1, \cdots, r$. Thus we have

$$\|(S_j^n - T)y_k\| \leq \|(S_j^n - T_j)x_{i_k}\| + \|(S_j^n - T_j)(x_{i_k} - y_k)\|$$

$$+ \|(T_j - T)y_k\| < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\bigcup_{n \geq 1} M_n$ is strongly dense in $\mathcal{R}$, and $\mathcal{R}$ is hyperfinite. The proof is complete.

Lemma 3.2. Let $\mathcal{R}$ be a countably decomposable factor of type II_∞ acting on a Hilbert space κ. Let $\mathfrak{M}$ be the ideal of definition of the trace, and let T be a positive operator in $\mathfrak{M}^{\frac{1}{2}}$. Let $\varepsilon > 0$ and $x_1, \cdots, x_r \in \kappa$. Then there is a finite type I subfactor M of $\mathcal{R}$ and $S \in M^+$ such that $\|S\| \leq \|T\|$ and $\|(S-T)x_k\| < \varepsilon$, $k = 1, \cdots, r$.

Proof. We first assume $T = E$ is a finite projection. Then given $\delta > 0$ there is an infinite projection G with infinite orthogonal complement such that $\|(G-E)x_k\| < \delta$. Indeed, let F be an infinite projection such that $F \geq E$ and such that $I - F$ is infinite. Considering $F\mathcal{R}F$ it thus suffices to find an infinite projection G such that $\|(G-E)x_k\| < \delta$. Since E is finite there is a finite type II_1-factor $\mathfrak{A}$ and a spatial isomorphism of $\mathcal{R}$ onto $\mathfrak{A} \otimes \mathcal{B}(\kappa)$ such that E is carried onto a projection $I \otimes P$ with P a one-dimensional projection in $\mathcal{B}(\kappa)$. Now there is a net $\{G_\alpha\}$ of infinite projections in $\mathcal{B}(\kappa)$ which converges strongly to P. Thus $I \otimes G_\alpha \to I \otimes P$ strongly, hence we can find an infinite projection G in $\mathcal{R}$ such that $\|(G-E)x_k\| < \delta$, $k = 1, \cdots, r$. From our reduction we can also assume $I - G$ is infinite.

Next assume $T = \sum_{i=1}^{n} \lambda_i E_i$ with $\lambda_i \geq 0$ and E_i orthogonal finite projections and $T \neq 0$. Let $F_i \geq E_i$, $i=1, \cdots, n$, be orthogonal infinite projections with sum I. By the above there exist infinite projections $G_i \leq F_i$ such that $F_i - G_i$ is infinite and $\|(G_i - E_i)x_k\| < \frac{\varepsilon}{2n}\|T\|^{-1}$, $k=1, \cdots, r$, $i=1, \cdots, n$. Let $G_{n+1} = \sum_{i=1}^{n} (F_i - G_i)$. Then $\sum_{i=1}^{n+1} G_i = I$ and all G_i are infinite. Furthermore we have

$$\left\|\left(\sum_{i=1}^{n} \lambda_i G_i - \sum_{i=1}^{n} \lambda_i E_i\right)x_k\right\| \leq \sum_{i=1}^{n} \lambda_i \|(G_i - E_i)x_k\| < \|T\| \sum_{i=1}^{n} \frac{\varepsilon}{2n} \|T\|^{-1} = \frac{\varepsilon}{2} .$$

Finally let T be a general positive operator in $\mathfrak{M}^{\frac{1}{2}}$ and $T \neq 0$. By spectral theory there is an operator $T_1 = \sum_{i=1}^{n} \lambda_i E_i$ with $\lambda_i > 0$ such that $0 \leq T_1 \leq T$ and $\|T - T_1\| < \varepsilon/2\max\{\|x_k\| : k=1,\cdots,r\}$. Since $T_1 \leq T$, E_i is finite for each i. By the last paragraph there are infinite orthogonal projections G_i, $i=1,\cdots,n+1$, such that if $S = \sum_{i=1}^{n} \lambda_i G_i$ then $\|S\| \leq \|T\|$, and $\|(S - T_1)x_k\| < \frac{\varepsilon}{2}$, $k=1,\cdots,r$. Thus we have

$$\|(S-T)x_k\| \leq \|(S-T_1)x_k\| + \|(T_1-T)x_k\| < \frac{\varepsilon}{2} + (\varepsilon/2\max\|x_k\|)\|x_k\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\mathcal{R}$ is countably decomposable the projections G_i are all equivalent, [1, Ch.III, § 8, Cor. 5]. Let V_i be a partial isometry in $\mathcal{R}$ such that $V_i V_i^* = G_i$, and $V_i^* V_i = G_1$, $i=2,\cdots,n+1$. Let $V_1 = G_1$. Let $e_{ij} = V_i V_j^*$. Then it is easy to see that the e_{ij}, $i,j = 1,\cdots,n+1$, form a complete set of matrix units for a type I_{n+1}-factor M containing all the G_i and hence S. This completes the proof of the lemma.

Remark 3.3. If $\mathcal{R}$ is a finite factor $\mathfrak{M}^{\frac{1}{2}} = \mathcal{R}$, and the lemma still holds. In this case the proof follows from spectral theory as above, the proof of [1, Ch.III, § 7, Lemme 8] and the equivalence of $\|\ \|_2$-convergence and strong convergence on bounded sets [1, Ch. III, § 7, Lemme 1].

A hyperfinite factor $\mathcal{R}$ is said to be an ITPFI-factor (= infinite tensor product of finite type I factors) if there is an infinite sequence of type I_{n_i}-factors M_i with $n_i \geq 2$ for an infinite number of i's and a product state $\omega = \bigotimes_{i=1}^{\infty} \omega_i$ of the C^*-algebra tensor product $\mathfrak{A} = \bigotimes_{i=1}^{\infty} M_i$, such that $\mathcal{R}$ equals the weak closure of $\pi(\mathfrak{A})$, where π is the cyclic representation of $\mathfrak{A}$ defined by ω. We also denote $\mathcal{R}$ by $\bigotimes_{i=1}^{\infty}(M_i,\omega_i)$.

The next lemma is known [5, §4].

Lemma 3.4. Let $\mathcal{R}$ be a factor of type II_∞. Let $\mathcal{F}$ be a hyperfinite II_1-factor, and let κ be a separable Hilbert space. Then $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\kappa)$ if and only if $\mathcal{R}$ is *-isomorphic to an ITPFI-factor.

Proof. Suppose $\mathcal{R}$ is *-isomorphic to an ITPFI-factor. In order to show $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\kappa)$ we may assume $\mathcal{R}$ is an ITPFI-factor, say $\mathcal{R} = \bigotimes_{i=1}^{\infty}(M_i,\omega_i)$. From a theorem of Takenouchi [5] there is a projection $P_i \in M_i$ such that if $Q_n = P_1 \otimes \cdots \otimes P_n \otimes I \otimes \cdots$ then the sequence $\{\pi(Q_n)\}$ converges strongly to a nonzero finite projection Q in $\mathcal{R}$. We show that $Q\mathcal{R}Q$ is hyperfinite. Let $N_n = \pi(\bigotimes_{i=1}^{n} M_i)$, where we consider $\bigotimes_{i=1}^{n} M_i$ as a subalgebra of $\mathfrak{A} = \bigotimes_{i=1}^{\infty} M_i$. Then $\{N_n\}$ is an increasing sequence of finite type I factors whose union is strongly dense in $\mathcal{R}$. We clearly have $\pi(Q_n)N_n\pi(Q_n) \cong QN_nQ$. Thus $\{QN_nQ\}$ is an increasing sequence of finite type I factors whose union is strongly dense in $Q\mathcal{R}Q$. But then $Q\mathcal{R}Q$ is hyperfinite so is *-isomorphic to $\mathcal{F}$ [1, Ch.III, § 7, Théorème 3]. Thus $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\kappa)$.

Conversely, if $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\kappa)$ then the above argument shows that $\mathcal{R}$ is *-isomorphic to an ITPFI-factor. The proof is complete.

4. Product factors. In this section we shall prove our main results on product factors. We shall only be concerned with semi-finite factors, and since all type I factors are well known, we shall only investigate factors of type II. Let $\mathcal{R}$ be a semi-finite factor. Let τ be a normal semi-finite trace on $\mathcal{R}$ and let $\mathfrak{m}$ be the ideal of definition of τ. With the inner product $(S,T) = \tau(T^*S)$ $\mathfrak{m}^{\frac{1}{2}}$ becomes a pre-Hilbert space. Let κ be its closure. Then the representation π of $\mathcal{R}$ on κ given by $\pi(S)R = SR$ for $R \in \mathfrak{m}^{\frac{1}{2}}$ is a $*$-isomorphism of $\mathcal{R}$ onto a factor acting on κ, see [1, Ch.I, §6, Théorème 2]. If we consider $\mathcal{R}$ in this representation we shall say $\mathcal{R}$ acts on the Hilbert space closure of $\mathfrak{m}^{\frac{1}{2}}$.

Lemma 4.1. Let $\mathcal{R}$ be a product factor of type II. Let τ be a normal semi-finite trace on $\mathcal{R}$ and let $\mathfrak{m}$ be the ideal of definition of τ. Consider $\mathcal{R}$ as acting on the Hilbert space closure of $\mathfrak{m}^{\frac{1}{2}}$. Let $A \in \mathfrak{m}^+$ and let M be a finite type I subfactor of $\mathcal{R}$. Let $x_1,\cdots,x_r$ be vectors in $\mathfrak{m}^{\frac{1}{2}}$ and let $\varepsilon > 0$. Then there is a finite type I factor N such that $M \subset N \subset \mathcal{R}$ and a positive operator $B \in N$ such that $\|(B - A)x_k\| < \varepsilon$, $k = 1,\cdots,r$.

Proof. Since $A \in \mathfrak{m}^+$ so is A^2. Multiplying A by a scalar we may assume $\tau(A^2) = 1$. Since $x_k \in \mathfrak{m}^{\frac{1}{2}}$ the vector state ω_{x_k} is of the form $\omega_{x_k}(C) = \tau(H_kC)$, where $H_k \in \mathfrak{m}^+$. Let $\delta = \min\{\varepsilon^2/4\|H_k\| : k = 1,\cdots,r\}$. Let ω be the normal state $\omega(C) = \tau(A^2C)$. Since $\mathcal{R}$ is a product factor, ω is asymptotically a product state. Hence there is a finite type I factor P such that $M \subset P \subset \mathcal{R}$ and such that

$$\|\omega - \omega|P \otimes \omega|P^c\| < \delta,$$

where $P^c = P' \cap \mathcal{R}$ (we follow here the notational convenience used in [4] of identifying $\mathcal{R}$ with $P \otimes P^c$). Since τ factors between P and P^c there are positive operators $S \in P$ and $T \in P^c$ such that

$$(\omega|P \otimes \omega|P^c)(C) = \tau((S^2 \otimes T^2)C)$$

for $C \in \mathcal{R}$. By [1, Ch.I, § 6, Théorème 8] if $H \in \mathcal{M}$ and ρ is the linear functional defined by $\rho(C) = \tau(HC)$ then $\|\rho\| = \|H\|_1$. Thus we have

$$\|A^2 - S^2 \otimes T^2\|_1 < \delta.$$

By Lemma 2.1 we then have

$$\|(A - S \otimes T)^2\|_1 = \|A - S \otimes T\|_2^2 \leq \|A^2 - S^2 \otimes T^2\|_1 < \delta.$$

Thus we have

$$\|(A - S\otimes T)x_k\| = \omega_{x_k}((A - S \otimes T)^2)^{\frac{1}{2}} = \tau(H_k(A - S\otimes T)^2)^{\frac{1}{2}}$$
$$< \|H_k\|^{\frac{1}{2}}\delta^{\frac{1}{2}} \leq \frac{\varepsilon}{2}.$$

Now P^c is a factor of the same type as $\mathcal{R}$ is, and $I \otimes T^2$ belongs to the ideal of definition of the trace on P^c. Thus by Lemma 3.2 if $\mathcal{R}$ is of type II_∞, and by Remark 3.3 if $\mathcal{R}$ is of type II_1, there is a finite type I factor $R \subset P^c$ and $T' \in R$ such that

$$\|(I \otimes T - I \otimes T')x_k\| < \frac{\varepsilon}{2}\|S\|^{-1}.$$

Let $B = S \otimes T'$ and let $N = P \otimes R$. Then N is a finite type I factor, $M \subset N \subset \mathcal{R}$, and $B \in N$. Furthermore we have

$$\|(A-B)x_k\| \leq \|(A - S\otimes T)x_k\| + \|(S\otimes T - S\otimes T')x_k\|$$
$$< \frac{\varepsilon}{2} + \|S\| \, \|(I\otimes T - I\otimes T')x_k\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete.

Lemma 4.2. Let $\mathcal{R}$ be as in Lemma 4.1. Let $S \in \mathfrak{m}^+$ with $\|S\| \leq 1$, and let M be a finite type I subfactor of $\mathcal{R}$. Then if $\varepsilon > 0$ and $y_1, \cdots, y_r \in \kappa$ we can find a finite type I factor N such that $M \subset N \subset \mathcal{R}$ and a positive operator $T \in N$ with $\|T\| \leq 1$ such that $\|(T - S)y_k\| < \varepsilon$, $k = 1, \cdots, r$.

Proof. In the proof we shall use the ideas used in the proof of the Kaplansky density theorem as it is given in [1, Ch.I, § 3, Théorème 3]. Since $\mathfrak{m}^{\frac{1}{2}}$ is dense in κ we can find $x_1, \cdots, x_r$ in $\mathfrak{m}^{\frac{1}{2}}$ such that $\|x_k - y_k\| < \frac{\varepsilon}{4}$. Since the function $x \to 2x(1 + x^2)^{-1}$ is strictly increasing from $[0,1]$ into $[0,1]$ we can find $A \in \mathcal{R}^+$ such that $S = 2A(I + A^2)^{-1}$. Then $A = \frac{1}{2}S(I + A^2)$, so

$$\tau(A) = \tfrac{1}{2}\,\tau(S(I + A^2)) \leq \tfrac{1}{2}\,\|I + A^2\|\,\tau(S) < \infty,$$

hence $A \in \mathfrak{m}^+$. Since $x_k \in \mathfrak{m}^{\frac{1}{2}}$ so are $(I + A^2)^{-1}x_k$ and Sx_k. We can thus apply Lemma 4.1 to A and find a finite type I factor N such that $M \subset N \subset \mathcal{R}$, and $B \in N^+$ such that

$$\|(B - A)(I + A^2)^{-1}x_k\| < \frac{\varepsilon}{8},$$

$$\|(B - A)Sx_k\| < \frac{\varepsilon}{2}.$$

Let $T = 2B(I + B^2)^{-1}$. Then $0 \leq T \leq I$, and $T \in N$ since $B \in N$. Moreover from [1, p.47] we have the identity

$$T - S = 2(I + B^2)^{-1}(B - A)(I + A^2)^{-1} + \tfrac{1}{2}T(A - B)S.$$

Since $\|(I + B^2)^{-1}\| \leq 1$ and $\|T\| \leq 1$ we have

$$\|(T - S)x_k\| \leq 2\|(I + B^2)^{-1}\|\,\|(B - A)(I + A^2)^{-1}x_k\| + \tfrac{1}{2}\|T\|\,\|(A - B)Sx_k\|$$

$$< 2\frac{\varepsilon}{8} + \frac{1}{2}\,\frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Thus we have

$$\|(T - S)y_k\| \leq \|(T - S)(y_k - x_k)\| + \|(T - S)x_k\|$$

$$< 2\ \|y_k - x_k\| + \frac{\varepsilon}{2}$$

$$< 2\ \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete.

Theorem 4.3. Let $\mathcal{R}$ be a factor of type II. Let $\mathcal{F}$ be a hyperfinite II_1-factor, and let $\mathcal{B}(\mathcal{H})$ be the bounded operators on a separable Hilbert space $\mathcal{H}$. Then we have:

i) If $\mathcal{R}$ is of type II_1 then $\mathcal{R} \cong \mathcal{F}$ if and only if $\mathcal{R}$ is a countably generated product factor.

ii) If $\mathcal{R}$ is of type II_∞ then $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\mathcal{H})$ if and only if $\mathcal{R}$ is a countably generated product factor.

Proof. If $\mathcal{R} \cong \mathcal{F}$ or $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\mathcal{H})$ then $\mathcal{R}$ is an ITPFI-factor, see Lemma 3.4. In particular $\mathcal{R}$ is countably generated, and by [4] $\mathcal{R}$ is a product factor. In order to show the converse we may assume $\mathcal{R}$ is a product factor acting on the Hilbert space closure κ of $\mathfrak{m}^{\frac{1}{2}}$, where $\mathfrak{m}$ is the ideal of definition of the trace on $\mathcal{R}$. Let M be a finite type I subfactor of $\mathcal{R}$, let $A \in \mathcal{R}^+$, $\|A\| \leq 1$, let $x_1, \cdots, x_r \in \kappa$, and let $\varepsilon > 0$. Since $\mathfrak{m}$ is a $*$-algebra which is strongly dense in $\mathcal{R}$ we may by the Kaplansky density theorem find $S \in \mathfrak{m}^+$ with $\|S\| \leq 1$ such that $\|(S - A)x_k\| < \frac{\varepsilon}{2}$. By Lemma 4.2 we can find a finite type I factor N such that $M \subset N \subset \mathcal{R}$ and

$T \in N$ such that $0 \leq T \leq I$, and $\|(T - S)x_k\| < \frac{\varepsilon}{2}$. Thus $\|(T - A)x_k\| < \varepsilon$, $k = 1,\cdots,r$. By Lemma 3.1 $\mathcal{R}$ is hyperfinite, hence by [4] $\mathcal{R}$ is *-isomorphic to an ITPFI-factor. If $\mathcal{R}$ is finite it is isomorphic to $\mathcal{F}$ (since all hyperfinite II_1-factors are *-isomorphic [1, Ch. III, § 7, Théorème 3]). If $\mathcal{R}$ is of type II_∞ $\mathcal{R} \cong \mathcal{F} \otimes \mathcal{B}(\mathcal{H})$ by Lemma 3.4. The proof is complete.

<u>Remark 4.4.</u> If $\mathcal{R}$ is finite in the above theorem it is possible to give a short direct proof using only Lemma 2.1. In another formulation the theorem states that if $\mathcal{R}$ is semi-finite but not finite dimensional, then $\mathcal{R}$ is *-isomorphic to an ITPFI-factor if and only if $\mathcal{R}$ is a countably generated product factor. It is an open question whether this is true if $\mathcal{R}$ is of type III.

<u>Corollary 4.5.</u> Let $\mathcal{R}$ be a countably generated product factor. If E is a nonzero projection in $\mathcal{R}$ then $E\mathcal{R}E$ is a countably generated product factor.

<u>Proof.</u> Since the map $A \to EAE$ of $\mathcal{R}$ into $E\mathcal{R}E$ is strongly continuous and surjective it is clear that $E\mathcal{R}E$ is countably generated. If E is an infinite projection then E is equivalent to I [1, Ch. III, § 8, Cor. 5], hence $E\mathcal{R}E$ is *-isomorphic to $\mathcal{R}$, so is a product factor. If E is a finite projection there are two cases. If $\mathcal{R}$ is of type I then $E\mathcal{R}E$ is a finite type I factor, so is trivially a product factor. If $\mathcal{R}$ is of type II, then by Theorem 4.3 $E\mathcal{R}E$ is a hyperfinite II_1-factor, since $E\mathcal{F}E$ is hyperfinite when $\mathcal{F}$ is the hyperfinite II_1-factor [1, Ch. III, § 7, Prop. 3]. This completes the proof of the corollary.

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